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Dimension of a Family of Singular Bernoulli Convolutions

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Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of i.i.d. Bernoulli random variables (i.e., X_n takes values $\{0, 1\}$ with probability $\frac{1}{2}$ each), let $X = \sum_{n=0}^{\infty} \rho^n X_n$, and let μ be the corresponding probability measure. Erdős-Salem proved that if $\frac{1}{2} < \rho < 1$, and if ρ^{-1} is a P.V. number, then μ is singular. In this paper, we study the algebraic structure of ρ and the singularity of the correspondent μ in more detail. We introduce a new class of algebraic numbers containing the P.V. numbers, and make use of the self-similar property determined by such numbers to calculate the exact *mean-quadratic-variation dimension* of μ. This dimension is most relevant to Strichartz's recent work on Fourier asymptotics of fractal measures.

1. INTRODUCTION

Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of i.i.d. Bernoulli random variables (i.e., X_n takes values $\{0, 1\}$ with probability $\frac{1}{2}$ each). For $0 < \rho < 1$, let $X = (1-\rho) \sum_{n=0}^{\infty} \rho^n X_n$, and μ the corresponding probability distribution, then μ is supported by $[0, 1]$ and is the infinite convolution of the sequence $\{\mu_n\}_{n=0}^{\infty}$, where μ_n is the point mass measure concentrated at 0 and $(1-\rho)\rho^n$, with weights $\frac{1}{2}$ each. Following the notation of Alexander and Yorke [AY], we call such μ an *infinitely convolved Bernoulli measure* (ICBM). It is known that if $0 < \rho < \frac{1}{2}$, then μ is a Cantor-type measure with Hausdorff dimension

$$\dim_{\text{Haus}}(\mu) := \inf\{\dim_{\text{Haus}}(E) : \mu(E) = 1\} = |\ln 2 / \ln \rho|. \tag{1.1}$$

If $\rho = \frac{1}{2}$, then μ is the Lebesgue measure restriction on $[0, 1]$. For $\frac{1}{2} < \rho < 1$, the situation is completely different: it was a conjecture in the 1930's that such μ should be absolutely continuous. This was disproved by Erdős [E1]; it is then known that if $\rho = 2^{1/k}$, $k = 2, 3, 4, \ldots$, or for almost all ρ sufficiently close to 1, then ρ is absolutely continuous [E2]. More

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fascinating results are known if $\beta=\rho^{-1}$ are algebraic integers: Let $\beta_1, \ldots, \beta_m$ denote the algebraic conjugates of β, then

(a) β is a P.V. number (i.e., $|\beta|>1$, and $|\beta_i|<1$, $i=1, \ldots, m$) if and only if $\hat{\mu}(\xi) \nrightarrow 0$ as $\xi \to \infty$. In particular μ is singular [S].

(b) If $\beta \prod_{|\beta_i|>1} \beta_i = 2$, then μ is absolutely continuous. Note that in this case $|\beta_i|>1$ for $i=1, \ldots, m$, necessarily [G].

There is still no satisfactory classification of ρ for μ to be singular or absolutely continuous. For $0<\rho<\frac{1}{2}$, one can easily determine $\dim_{\text{Haus}}(\mu)$, the "degree of singularity" of μ, as in (1.1); for $\frac{1}{2}<\rho<1$, with $\rho^{-1}=\beta$ a P.V. number, it is difficult to evaluate such a dimension exactly (see [AY] for the case $\rho=(\sqrt{5}-1)/2$, and also [PU]). Recently another type of dimensional index for fractal measures is being studied; it is natural, easy to calculate, and adapts well for Fourier analysis ([L, LW; Str1–4]). Our main purpose in this paper is to study such a dimension for the ICBM derived from a class of algebraic integers $\beta=\rho^{-1}$ which contains the P.V. numbers in (a).

For a bounded nonnegative measure μ on $\mathbb{R}^d$, we let

$$\Phi^{(\alpha)}(h)=\frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} \mu(Q_h(x))^2\, dx,$$

where $Q_h(x)$ is the half open cube $\prod_{j=1}^d (x_i-h, x_i+h]$. It follows from a theorem of Hardy and Littlewood [HL] that $\sup_{h>0} \Phi^d(h)<\infty$ if and only if μ is absolutely continuous with $d\mu/dx=g$ in $L^2(\mathbb{R}^d)$, and

$$\lim_{h\to 0} \Phi^{(d)}(h)=\int_{\mathbb{R}^d} g^2.$$

We define the *α-mean quadratic variation* (m.q.v.) of μ as $\limsup_{h\to 0} \Phi^\alpha(h)$, and the *m.q.v. dimension* of μ as

$$\dim_{\text{m.q.v.}}(\mu)=\inf\{\alpha : 0<\limsup_{h\to 0} \Phi^{(\alpha)}(h)\}. \tag{1.2}$$

Note that the set of α above is non-empty as it always contains d (for otherwise, $\limsup_{h\to 0} \Phi^{(d)}(h)=0$ implies that $\sup_{h>0} \Phi^{(d)}(h)<\infty$. By the above remark, $d\mu/dx=g$ for some square integrable g on $\mathbb{R}^d$, and

$$0=\lim_{h\to 0} \Phi^{(\alpha)}(h)=\int_{\mathbb{R}^d} g^2.$$

Hence $\mu=0$ and is a contradiction). It is clear that if $\beta=\dim_{\text{m.q.v.}}(\mu)$, then

$$\limsup_{h\to 0} \Phi^{(\alpha)}(h)=\begin{cases} 0 & \text{if } \alpha<\beta \\ \infty & \text{if } \alpha>\beta. \end{cases}$$

Also $\dim_{\text{m.q.v.}}(\mu) < d$ implies that μ is singular. For the two extreme cases: if μ is absolutely continuous, then $\dim_{\text{m.q.v.}}(\mu) = d$; if μ is a discrete measure, then $\dim_{\text{m.q.v.}}(\mu) = 0$. The mean quadratic variation of fractal measures together with its Fourier transformation has been studied in detail in [L, LW, Str1–4]; there are identities and inequalities analogous to the Plancherel identity. Some of the ideas developed there can be traced back to Besicovitch's almost periodic functions, and Wiener's generalized harmonic analysis [W], where they dealt with the case $\alpha = 0$ for some class of distributions.

Recall that a probability measure μ on $\mathbb{R}^d$ is called a *self-similar measure* [H] if μ satisfies

$$\mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1}, \tag{1.3}$$

where $S_j(x) = \rho_j R_j x + b_j$ with $0 < \rho_j < 1$, R_j rotations on $\mathbb{R}^d$, $b_j \in \mathbb{R}^d$, and $a_j > 0$, $a_1 + \cdots + a_m = 1$. For $S_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, with

$$S_1(x) = \rho x, \quad S_2(x) = \rho x + (1 - \rho), \qquad x \in \mathbb{R}, \tag{1.4}$$

and $a_1 = a_2 = \frac{1}{2}$, the self-similar measure equals the ICBM described above [L, Theorem 4.3]. For the case $0 < \rho < \frac{1}{2}$, (1.1) has been extended as follows ([LW, Str4]):

If $\{S_j\}_{j=1}^m$ satisfies the open set condition with an open set U such that $\mu(bdry U) = 0$, then $\dim_{\text{m.q.v.}}(\mu) = \alpha$ where α is given by $\sum_{j=1}^m a_j \rho_j^{-\alpha} = 1$. Moreover the mean quadratic average $q(r) = r^{-(1-\alpha)} \int_{-r}^{r} |\hat{\mu}|^2$ of the Fourier transformation $\hat{\mu}$ is asymptotically multiplicative period.

For the case $\frac{1}{2} < \rho < 1$ ($\{S_i\}_{i=1}^2$ does not satisfy the open set condition), only the case $\rho = (\sqrt{5} - 1)/2$ has been considered [L]. By reducing the expression of the mean quadratic variation to a functional equation and making use of the renewal equation in probability theory, it is proved that *the m.q.v. dimension* α of μ satisfies

$$(4\rho^\alpha)^3 - 2(4\rho^\alpha)^2 - 2(4\rho^\alpha) + 2 = 0 \qquad (\alpha = 0.9923994...).$$

Note that the above formulas determining the dimensions in the two cases are significantly different. In the following we continue on the case $\frac{1}{2} < \rho < 1$. For $1 < \beta < 2$, we let $s^{(0)} = 0$, $s^{(n+1)} = \beta s^{(n)} + \varepsilon_n$, $\varepsilon_n = 0, 1$, or -1. We call an algebraic number β an *F-number* if the set

$$W_\beta = \{s^{(n)} : |s^{(n)}| < 1/(\beta - 1), n \in \mathbb{N}\}$$

is a finite set. This class of numbers contains all P.V. numbers (Theorem 2.5), in particular $(\sqrt{5} + 1)/2$, and is contained in the class of

Beta numbers of Renyi [P, Proposition 2.6]. Our main theorem is the following (Theorem 4.2):

THEOREM A. *Suppose* $\frac{1}{2} < \rho < 1$, *and* $\rho^{-1} = \beta$ *is an F-number. Let* μ *be the ICBM determined by* ρ *as in* (1.4), *then* μ *is singular. Moreover* $\alpha = \dim_{\text{m.q.v.}}(\mu)$ (<1) *is given by* $4\rho^{\alpha} = \lambda$, *where* λ *is the maximal eigenvalue of some non-negative matrix A determined by* β.

The main idea of the proof is to apply (1.3) repeatedly to the right-hand side of

$$\Phi^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_{-\infty}^{\infty} \mu(Q_h(x))^2 \, dx.$$

(α is to be determined.) This iteration process yields a functional equation in terms of a linear operator T over the vector space generated by the elements of W_β considered as "words" (see (3.6)). The finiteness of W_β then allows an explicit matrix representation A of T, the maximal eigenvalue, and hence the m.q.v. dimension can be calculated.

Once the dimension is known, the Fourier transformation $\hat{\mu}$ of the ICBM determined by the F-numbers behaves similarly to the general case of self-similar measures with the open set condition (Theorem 4.8).

THEOREM B. *Let* μ *and* $\alpha = \dim_{\text{m.q.v.}}(\mu)$ *be as above, and let*

$$q(r) = \frac{1}{r^{1-\alpha}} \int_{-r}^{r} |\hat{\mu}|^2.$$

Suppose A is irreducible, then $q(r) \not\equiv 0$ *is an asymptotically multiplicative periodic function of period* ρ, i.e., $\lim_{r\to\infty} (q(r) - \tilde{q}(r)) = 0$, *where* $\tilde{q}(\rho r) = \tilde{q}(r)$ *for* $r > 0$.

The theorem hence provides different information on $\hat{\mu}$ compared to that given by Erdős-Salem's result in (a).

We organize the paper as follows: in Section 2 we introduce a family of random paths generated by β and the subset W_β of the states of the paths. The class of F-numbers is defined in terms of W_β; it contains all P.V. numbers, and is contained in the class of Beta numbers. In Section 3 we use the self-similar property and the mean quadratic variation to set up a dynamic on a set of lines Δ considered as symbols. The dynamic on Δ is represented as a map T in (3.6) and is used to obtain a functional equation (Proposition 3.3) which the theory is based upon. The set of relevant symbols is identified with W_β. In Section 4 we make use of the Perron–

Frobenius theorem on non-negative matrices and a functional equation (Proposition 3.6) to prove Theorems A and B. Finally, in Section 5, we give some numerical examples and discuss some unsettled relationships of the class of F-numbers with the P.V. numbers, Beta numbers, and Salem numbers.

2. Some Algebraic Numbers

For $\beta > 1$, we define a family of "random paths" as follows: let $s^{(0)} = \varepsilon_0 = 0$ be the initial state, for $(\varepsilon_1, ..., \varepsilon_n)$, $\varepsilon_j = 0$ or ± 1, $1 \leqslant j \leqslant n$, let

$$s^{(n)} = \beta s^{(n-1)} + \varepsilon_n = \sum_{j=0}^{n-1} \varepsilon_{n-j} \beta^j$$

be the path at the nth stage. For any two states s, s', we write $s \lhd s'$ to mean that there exist $(\varepsilon_1, ..., \varepsilon_n)$ such that $s = s^{(k)}$ and $s' = s^{(n)}$, $1 \leqslant k < n$ (i.e., s' is connected to s via $(\varepsilon_{k+1}, ..., \varepsilon_n)$); they are called *bi-connected* if $s \lhd s'$ and $s' \lhd s$.

Proposition 2.1. *For $\beta > 1$, let $M = 1/(\beta - 1)$. Suppose $s \lhd s'$, then*

(i) *if $|s'| < M$, then $|s| < M$,*

(ii) *if s' is bi-connected to 0, then s is also bi-connected to 0. In particular, both s and s' are bounded by M.*

Proof. Suppose $|s| \geqslant M$. Let $s = s^{(k)}$, $s' = s^{(n)}$, with $k < n$, then

$$|s^{(k+1)}| = |\beta s^{(k)} + \varepsilon_{k+1}| \geqslant \beta \cdot M - 1 = M.$$

Inductively, we have $|s^{(n)}| \geqslant M$ and (i) follows.

The first assertion in (ii) is a direct consequence of the definition of bi-connectedness. The second assertion follows from (i).

Let

$$W_{0,\beta} = \left\{ s = \sum_{j=0}^{n-1} \varepsilon_{n-j} \beta^j \colon \varepsilon_j = 0 \text{ or } \pm 1,\ 1 \leqslant j \leqslant n,\ n \in \mathbb{N} \right\}$$

be the states of all the paths, and let

$$W_\beta = \left\{ s \in W_{0,\beta} \colon |s| < \frac{1}{\beta - 1} \right\}.$$

It follows from Proposition 2.1 that if a path steps outside the interval $(-1/(\beta-1), 1/(\beta-1))$, then it will never return.

PROPOSITION 2.2. *Suppose $\beta > 1$ is an algebraic number with minimal polynomial $p(x)$. Let $E = \{s \in W_\beta : s$ is bi-connected to $0\}$. If $E \neq \emptyset$, then $\# E \geqslant 2(\deg p) + 1$.*

Proof. Since $E \neq \emptyset$, there exists a path that starts at 0, passes through some $s \in E$, and ends at 0; i.e., $\sum_{j=0}^{n-1} \varepsilon_{n-j}\beta^j = 0$ for some $(\varepsilon_1, ..., \varepsilon_n)$. Let m be the smallest of all such integers n, and let $(\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_m)$ be the corresponding vector defining the state $s^{(m)}$ $(=0)$. Let

$$E_0 = \left\{ s^{(k)} = \sum_{j=0}^{k-1} \tilde{\varepsilon}_{k-j}\beta^j : 0 \leqslant k \leqslant m-1 \right\}.$$

Then $E_0 \subseteq E$, and the minimality of m implies that all the elements of E_0 are distinct so that $\# E_0 = m$. Also by symmetry, $-E_0 \subseteq E$. Note that $-E_0 \cap E_0 = \{0\}$: for otherwise there exists $s^{(k)} = -s^{(k')} \neq 0$ for some $k' > k$, say; then the path defined by $(\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_m)$ from 0 to 0 can be shortened as $(\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_k, -\tilde{\varepsilon}_{k'+1}, ..., -\tilde{\varepsilon}_m)$ which contradicts the minimality of m.

Since $p(x)$ is the minimal polynomial of β, $p(x)$ divides $\sum_{j=0}^{m-1} \tilde{\varepsilon}_{k-j} x^j$, so that $(m-1) \geqslant \deg p$. It follows that

$$\# E \geqslant 2(\# E_0) - 1 = 2m - 1 \geqslant 2(\deg p) + 1.$$

(subtract 1 because 0 is used twice).

PROPOSITION 2.3. *Suppose $1 < \beta < 2$ is an algebraic number with minimal polynomial $p(x)$, then $\# (W_\beta) \geqslant 2(\deg p) + 1$.*

Proof. We need only consider the case $\# (W_\beta) < \infty$. Consider the map F defined by $F(x) = \beta x + \varepsilon$, where $\varepsilon = 0$ if $0 \leqslant \beta x < 1$, and $\varepsilon = -1$ if $\beta \geqslant 1$. Let $\tilde{\varepsilon}_0 = 1$, $\tilde{\varepsilon}_1 = -1$, by taking $x = 1$, we have $F(1) = \beta - 1$, $F^2(1) = \beta(\beta-1) + \tilde{\varepsilon}_2$, where $\tilde{\varepsilon}_2 = 0$ or -1, and

$$F^n(1) = \sum_{j=0}^{n} \tilde{\varepsilon}_{n-j}\beta^j \in [0, 1).$$

Let $(\varepsilon_1, ..., \varepsilon_{n+1}) = (\tilde{\varepsilon}_0, ..., \tilde{\varepsilon}_n)$, we see that $F^n(1)$ are states in W_β. Since W_β is a finite set by assumption, there exists m and n smallest such that $m > n$ and $F^m(1) = F^n(1)$, i.e.,

$$\sum_{j=0}^{m} \tilde{\varepsilon}_{m-j}\beta^j - \sum_{j=0}^{n} \tilde{\varepsilon}_{n-j}\beta^j = 0.$$

This implies that β is a root of the polynomial $q(x)$ of the above form, $p(x)$ hence divides $q(x)$ and $m \geqslant \deg p$. Also the minimality of m and n implies that all the $\sum_{j=0}^{k} \tilde{\varepsilon}_{k-j}\beta^j$, $0 \leqslant k < m-1$, in W_β are distinct. The number of non-negative elements of W_β is hence $\geqslant \deg p$, so that $\#(W_\beta) \geqslant 2(\deg p) - 1$.

We call an algebraic integer $1 < \beta < 2$ an *F-number* if W_β is a finite set. An algebraic integer β is called a *Pisot-Vijayarahavan (P.V.) number* if $\beta > 1$ and all its conjugates have moduli less than 1.

PROPOSITION 2.4. *Suppose $1 < \beta < 2$ is an F-number, let $E = \{s \in W_\beta : s$ is bi-connected to $0\}$. Then $E \neq \varnothing$, so that $\#(W_\beta) \geqslant \# E \geqslant 2(\deg p) + 1$.*

Proof. Let q, q' be any two polynomials of the form $\sum_{j=0}^{n} a_j x^j$, $a_j = 0$ or 1, then $q(\beta) - q'(\beta) = \sum_{j=0}^{n} \varepsilon_j \beta^j$, $\varepsilon_j = -1, 0$, or 1. It follows from the assumption that if $q(\beta) \neq q'(\beta)$, then

$$|q(\beta) - q'(\beta)| \geqslant \min\{|s| : s \in W_\beta, s \neq 0\} > 0.$$

Note that $|q(\beta)| \leqslant C\beta^n$. On the other hand there are 2^n combination's of such $q(\beta)$, hence there exist q, q' such that $q(\beta) - q'(\beta) = 0$. This implies that $E \neq \varnothing$. The rest of the statement follows from Proposition 2.2.

By using an idea similar to that in [Sc], we prove

THEOREM 2.5. *Suppose $1 < \beta < 2$ is a P.V. number, then β is an F-number.*

Proof. Let $p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$, $a_i \in \mathbb{Z}$, $i = 0, ..., d-1$, be the minimal polynomial for $\beta = \beta_1$, and let $\beta_2, ..., \beta_d$ be the conjugate roots. For each n, let

$$s_1^{(n)} = \sum_{j=0}^{n-1} \varepsilon_{n-j} \beta_1^j \in W_\beta.$$

By using $p(\beta_1) = 0$ and long division, we can write

$$s_1^{(n)} = \sum_{j=0}^{d-1} p_j^{(n)} \beta_1^j, \tag{2.1}$$

where $p_j^{(n)} \in \mathbb{Z}$. Note that β_1 is a root of $l(x) = \sum_{j=0}^{n-1} \varepsilon_{n-j} x^j = \sum_{j=0}^{d-1} p_j^{(n)} x^j$, so that $p(x)$ divides $l(x)$. Hence β_i, $i = 2, ..., n$, are also roots of $l(x)$. This implies that

$$s_k^{(n)} := \sum_{j=0}^{n-1} \varepsilon_{n-j} \beta_k^j = \sum_{j=0}^{d-1} p_j^{(n)} \beta_k^j, \qquad 1 \leqslant k \leqslant d, \tag{2.2}$$

which can be expressed in matrix form:

$$\begin{bmatrix} s_1^{(n)} \\ \vdots \\ s_d^{(n)} \end{bmatrix} = A \begin{bmatrix} p_0^{(n)} \\ \vdots \\ p_{d-1}^{(n)} \end{bmatrix}, \qquad \text{where} \quad A = \begin{bmatrix} 1 \cdots \beta_1^{(d-1)} \\ \vdots \qquad \vdots \\ 1 \cdots \beta_d^{(d-1)} \end{bmatrix}.$$

Since A is non-singular, we have

$$\begin{bmatrix} p_0^{(n)} \\ \vdots \\ p_{d-1}^{(n)} \end{bmatrix} = A^{-1} \begin{bmatrix} s_1^{(n)} \\ \vdots \\ s_d^{(n)} \end{bmatrix}.$$

It follows from the definition of W_β that $|s_1^{(n)}| < (\beta - 1)^{-1}$. Since β is a P.V. number, the conjugates β_k have moduli less than 1, and (2.2) implies that $|s_k^{(n)}| < (1 - \beta_k)^{-1}$, $k = 2, ..., d$. The left side of the above is hence bounded for all $n \in \mathbb{N}$. Consequently, $\{p_1^{(n)}: n \in \mathbb{N}\}$ is a bounded set of integers and must be finite. We thus conclude, by (2.1), that W_β is a finite set; i.e., β is an F-number.

For $\beta > 1$ the *Beta expansion* of x is defined as $x = \sum_{n=0}^{\infty} a_{n+1} \beta^{-n}$, where $a_{n+1} = [\beta F^n(x)]$ where F is defined as in the proof of Proposition 2.3. An algebraic integer $\beta > 1$ is called a *Beta number* if the a_n's in the Beta expansion of β are eventually periodic; i.e., there exists k and m such that for $n \geqslant m$, $a_n = a_{n+k}$; β is called a *Perron number* if $|\beta'| < \beta$ for any algebraic conjugate β' of β. It is known that Beta numbers are Perron numbers (e.g., [Li]).

PROPOSITION 2.6. *If β is an F-number, then β is a Beta-number and hence a Perron number.*

Proof. Let $\beta = \sum_{n=0}^{\infty} a_{n+1} \beta^{-n}$, where $a_{n+1} = [\beta F^n(\beta)]$. It follows from the proof of Proposition 2.3 that $F^m(1) = F^n(1)$ for some $m > n$. Let m, n be such that $k = m - n$ is smallest, then for any non-negative integer l

$$a_{n+l} = [\beta F^{n+l-1}(\beta)] = [\beta F^{n+l}(1)] = [\beta F^{n+k+l}(1)] = a_{n+k+l}.$$

This implies that $\{a_n\}$ is eventually periodic, β is hence a Beta number, and also a Perron number.

3. Mean Quadratic Variations

Throughout we assume that $\frac{1}{2} < \rho < 1$, and μ is the self-similar measure defined by $S_1(x) = \rho x$, $S_2(x) = \rho x + (1-\rho)$, $x \in \mathbb{R}$, and $a_1 = a_2 = \frac{1}{2}$. It follows from (1.3) that

$$\begin{aligned}\mu(E) &= \frac{1}{2}\,\mu(S_1^{-1}(E)) + \frac{1}{2}\,\mu(S_2^{-1}(E)) \\ &= \frac{1}{2}\left(\mu\left(\frac{E}{\rho}\right) + \mu\left(\frac{E}{\rho} - \frac{(1-\rho)}{\rho}\right)\right) \end{aligned} \tag{3.1}$$

for any Borel subset E in $\mathbb{R}$. Let Δ denote the class of lines γ with slope 1 and x-intercept at $a \in \mathbb{R}$; i.e.,

$$\gamma : \begin{cases} x = t + a \\ y = t \end{cases}, \qquad -\infty < t < \infty.$$

For $h > 0$, we let $\Phi_\gamma^{(\alpha)}(h)$ be the line integral defined by

$$\Phi_\gamma^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_\gamma \mu(Q_h(x))\,\mu(Q_h(y)), \tag{3.2}$$

where $Q_h = [x-h, x+h)$. It follows that

$$\Phi_\gamma^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_{-\infty}^{\infty} \mu(Q_h(t+a))\,\mu(Q_h(t))\,dt. \tag{3.3}$$

Note that μ is concentrated on a dense subset of [0, 1], and we see that the "effective domain" of integration in (3.2) is on $\gamma \cap ([0,1] \times [0,1])$ (see Fig. 1).

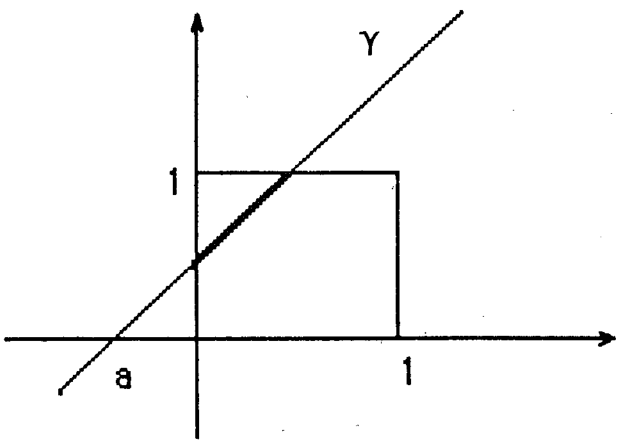

FIGURE 1

PROPOSITION 3.1. *Let $\gamma \in \Delta$ and has x-intercept at a, then*

(i) *if γ' has x-intercept $-a$, then $\Phi_\gamma^{(\alpha)}(h) = \Phi_{\gamma'}^{(\alpha)}(h)$ for all $h > 0$.*

(ii) *$a \notin [-1, 1]$ if and only if there exists $h_0 > 0$ such that $\Phi_\gamma^{(\alpha)}(h) = 0$ for $0 < h < h_0$.*

(iii) *For $a = 1$ or -1, there exists $\delta > 0$ such that for $0 < \alpha \leqslant 1$, $\Phi_\gamma^{(\alpha)}(h) = o(h^\delta)$ as $h \to 0$.*

Proof. (i) follows from a change of variable by $s = t + a$ in (3.3). To prove (ii), we use an argument which is clear from Fig. 1. If $a \notin [-1, 1]$, let $h_0 = (|a| - 1)/2$, then for $0 < h < h_0$, $-\infty < t < \infty$, either $Q_h(t) \cap [0, 1] = \varnothing$ or $Q_h(t + a) \cap [0, 1] = \varnothing$. Since μ is supported by $[0, 1]$, it follows that

$$\int_\gamma \mu(Q_h(x))\, \mu(Q_h(y)) = 0,$$

so that $\Phi_\gamma^{(\alpha)}(h) = 0$ for $0 < h < h_0$. On the other hand, if $a \in [-1, 1]$, then, for any h, there exists t_0 such that $[t_0 - h, t_0 + h)$ and $[t_0 + a - h, t_0 + a + h)$ intersect $[0, 1]$ (if $0 \leqslant a \leqslant 1$, take $t_0 = 0$, and if $-1 \leqslant a \leqslant 0$, take $t_0 = -a$). It follows that $\mu(Q_h(t_0))$ and $\mu(Q_h(t_0 + a))$ are both nonzero. This implies that

$$\int_\gamma \mu(Q_h(x))\, \mu(Q_h(y)) \neq 0,$$

and hence $\Phi_\gamma^{(\alpha)}(h) \neq 0$ for any $h > 0$.

For (iii) we assume that $a = 1$. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \mu(Q_h(t+1))\, \mu(Q_h(t)) &= \int_{-h}^{h} \mu(Q_h(t+1))\, \mu(Q_h(t)) \\
&\leqslant \left(\int_{-h}^{h} |\mu(Q_h(t+1))|^2 \right)^{1/2} \left(\int_{-h}^{h} |\mu(Q_h(t))|^2 \right)^{1/2} \\
&= \int_{-h}^{h} |\mu(Q_h(t))|^2 \leqslant 2 \int_0^h |\mu(Q_h(t))|^2.
\end{aligned}$$

(The second equality holds since μ is symmetric about $\frac{1}{2}$). Note that for $E \subseteq [0, 1 - \rho)$, (3.1) implies that $\mu(E) = \frac{1}{2}\, \mu(E/\rho)$. Hence

$$\begin{aligned}
\int_0^h |\mu(Q_h(t))|^2 &= \frac{1}{4} \int_0^h |\mu(Q_{h/\rho}(t/\rho))|^2 \\
&= \frac{\rho}{4} \int_0^{h/\rho} |\mu(Q_{h/\rho}(t))|^2 = \left(\frac{\rho}{4}\right)^n \int_0^{h/\rho^n} |\mu(Q_{h/\rho^n}(t))|^2,
\end{aligned}$$

provided that $h/\rho^n < 1-\rho$. Let $N = [(\ln(h/(1-\rho)))/\ln \rho]$, then

$$\int_0^h |\mu(Q_h(t))|^2 \leqslant \left(\frac{\rho}{4}\right)^N \int_0^{1-\rho} |\mu(Q_{h/\rho^N}(t))|^2$$
$$\leqslant \left(\frac{\rho}{4}\right)^N \leqslant \left(\frac{h}{1-\rho}\right)^{1-\ln 4/\ln \rho} = o(h^{2+\delta}),$$

for some $\delta > 0$. It follows that $\Phi_\gamma^{(\alpha)}(h) = o(h^\delta)$.

We consider Δ as a set of words and that it spans a real vector space $\langle \Delta \rangle$. According to the common convention of line integral, we have

$$\Phi^{(\alpha)}_{c_1\gamma_1 + \cdots + c_n\gamma_n}(h) = c_1 \Phi^{(\alpha)}_{\gamma_1}(h) + \cdots + c_n \Phi^{(\alpha)}_{\gamma_n}(h)$$

for any $c_1\gamma_1 + \cdots + c_n\gamma_n \in \langle \Delta \rangle$.

LEMMA 3.2. *Let $\gamma \in \langle \Delta \rangle$. Suppose that $\Phi_\gamma^{(\alpha)}(h) = 0$ for all $h > 0$, then*

$$\gamma = (c_1\gamma_1 + \cdots + c_n\gamma_n) - (c_1\gamma_1' + \cdots + c_n\gamma_n'),$$

where $\gamma_j, \gamma_j' \in \Delta$ have x-intercepts a_j, $-a_j$, $1 \leqslant j \leqslant n$, respectively.

Proof. Let $\gamma = \sum_{j=1}^m c_j\gamma_j$, then $\Phi_\gamma^{(\alpha)}(h) = 0$ implies that $\int_\gamma \mu(Q_h(x))\, \mu(Q_h(y)) = 0$, i.e.,

$$\sum_{j=1}^m c_j \int_{-\infty}^{\infty} \mu(Q_h(t + a_j))\, \mu(Q_h(t))\, dt = 0 \qquad \text{for all} \quad h > 0. \tag{3.4}$$

Denote $F(t) = \mu(-\infty, t]$, then

$$\int_{-\infty}^{\infty} \mu(Q_h(t))\, e^{i\xi t}\, dt = \int_{-\infty}^{\infty} (F(t+h) - F(t-h))\, e^{i\xi t}\, dt$$
$$= \frac{1}{-i\xi} \int_{-\infty}^{\infty} e^{i\xi t}\, d(F(t+h) - F(t-h))$$
$$= \frac{\sin h\xi}{\xi} \int_{-\infty}^{\infty} e^{i\xi t}\, dF(t)$$
$$= \hat{\mu}(\xi)\, \frac{\sin h\xi}{\xi}.$$

By Plancherel theorem, (3.4) reduces to

$$\sum_{j=1}^m c_j \int_{-\infty}^{\infty} |\hat{\mu}(\xi)|^2 \frac{\sin^2 h\xi}{\xi^2}\, e^{ia_j\xi}\, d\xi = 0 \qquad \text{for all} \quad h > 0. \tag{3.5}$$

Since μ is symmetric about $\frac{1}{2}$,

$$\int_{-\infty}^{\infty} e^{i\xi t}\, d\mu(t) = \int_{-\infty}^{\infty} e^{i\xi t}\, d\mu(1-t).$$

This implies that $\hat{\mu}(\xi) = e^{-i\xi}\hat{\mu}(-\xi)$, so that $|\hat{\mu}(\xi)| = |\hat{\mu}(-\xi)|$. We can rewrite (3.5) as

$$\int_0^{\infty} \left(\sum_{j=1}^{m} c_j e^{ia_j\xi} + \sum_{j=1}^{m} c_j e^{-ia_j\xi} \right) |\hat{\mu}(\xi)|^2 \frac{\sin^2 h\xi}{\xi^2}\, d\xi = 0 \qquad \text{for all} \quad h > 0.$$

Wiener's Tauberian theorem [T, Theorem 7.6] then implies that

$$\left(\sum_{j=1}^{m} c_j e^{ia_j\xi} + \sum_{j=1}^{m} c_j e^{-ia_j\xi} \right) |\hat{\mu}(\xi)|^2 = 0 \qquad \text{for all} \quad \xi \in \mathbb{R}.$$

This can only happen when $c_j = -c_k$, $a_j = -a_k$ for some $j \neq k$. By rearranging the indices, γ is of the form as stated.

Let $T: \langle \Delta \rangle \to \langle \Delta \rangle$ be the linear map defined by

$$T(\gamma) = \gamma^{(-1)} + 2\gamma^{(0)} + \gamma^{(1)}, \qquad \gamma \in \Delta, \tag{3.6}$$

where

$$\gamma^{(\varepsilon)}: \begin{cases} x = t + \left(\dfrac{a}{\rho} + \varepsilon \dfrac{1-\rho}{\rho} \right), & -\infty < t < \infty, \\ y = t \end{cases} \tag{3.7}$$

and $\varepsilon = -1$, 0, or 1. Note that $\gamma^{(\varepsilon)}$ has x-intercept $a/\rho + \varepsilon(1-\rho)/\rho$ (Fig. 2).

The most important property of T we use is the following:

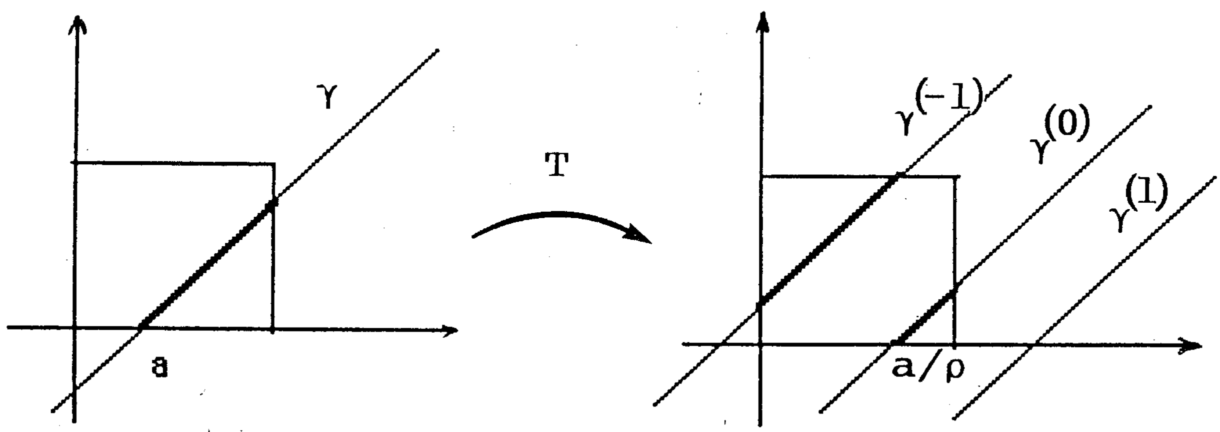

FIGURE 2

PROPOSITION 3.3. *For* $0 \leqslant \alpha \leqslant 1$, $\gamma \in \langle \Delta \rangle$, $h > 0$, *we have* $\Phi_\gamma^{(\alpha)}(h) = 1/4\rho^\alpha \, \Phi_{T\gamma}^{(\alpha)}(h/\rho)$.

Proof. By the linearity of T, we need only consider $\gamma \in \Delta$, i.e.,

$$\gamma : \begin{cases} x = t + a \\ y = t \end{cases}, \qquad -\infty < t < \infty.$$

It follows from (3.1) that

$$\Phi_\gamma^{(\alpha)}(h) = \frac{1}{4h^{1+\alpha}} \int_\gamma \left(\mu\left(Q_{h/\rho}\left(\frac{x}{\rho}\right) + \mu\left(Q_{h/\rho}\left(\frac{x}{\rho} - \frac{1-\rho}{\rho}\right)\right)\right)\right)$$
$$\times \left(\mu\left(Q_{h/\rho}\left(\frac{y}{\rho}\right) + \mu\left(Q_{h/\rho}\left(\frac{y}{\rho} - \frac{1-\rho}{\rho}\right)\right)\right)\right).$$

The integrand multiplies out to have four terms. First we have

$$\int_\gamma \mu\left(Q_{h/\rho}\left(\frac{x}{\rho}\right)\right) \mu\left(Q_{h/\rho}\left(\frac{y}{\rho}\right)\right) = \int_{-\infty}^{\infty} \mu\left(Q_{h/\rho}\left(\frac{t+a}{\rho}\right)\right) \mu\left(Q_{h/\rho}\left(\frac{t}{\rho}\right)\right) dt$$
$$= \rho \int_{-\infty}^{\infty} \mu\left(Q_{h/\rho}\left(t + \frac{a}{\rho}\right)\right) \mu(Q_{h/\rho}(t))\, dt$$
$$= \rho \int_{\gamma(0)} \mu(Q_{h/\rho}(x))\, \mu(Q_{h/\rho}(y)).$$

Similarly, we can show that

$$\int_\gamma \mu\left(Q_{h/\rho}\left(\frac{x}{\rho}\right)\right) \mu\left(Q_{h/\rho}\left(\frac{y}{\rho} - \frac{1-\rho}{\rho}\right)\right) = \rho \int_{\gamma(1)} \mu(Q_{h/\rho}(x))\, \mu(Q_{h/\rho}(y)),$$
$$\int_\gamma \mu\left(Q_{h/\rho}\left(\frac{x}{\rho} - \frac{1-\rho}{\rho}\right)\right) \mu\left(Q_{h/\rho}\left(\frac{y}{\rho}\right)\right) = \rho \int_{\gamma(-1)} \mu(Q_{h/\rho}(x))\, \mu(Q_{h/\rho}(y)),$$

and

$$\int_\gamma \mu\left(Q_{h/\rho}\left(\frac{x}{\rho} - \frac{1-\rho}{\rho}\right)\right) \mu\left(Q_{h/\rho}\left(\frac{y}{\rho} - \frac{1-\rho}{\rho}\right)\right)$$
$$= \rho \int_{\gamma(0)} \mu(Q_{h/\rho}(x))\, \mu(Q_{h/\rho}(y)).$$

Putting these together we have $\Phi_\gamma^{(\alpha)} = (1/4\rho^\alpha)\, \Phi_{T\gamma}^{(\alpha)}(h/\rho)$.

Let $\varepsilon_0 = 0$, and let $\gamma^0 = \gamma^{\varepsilon_0} \in \Delta$ be the line with x-intercept 0. Define $\gamma^{\varepsilon_0 \cdots \varepsilon_n} = (\gamma^{\varepsilon_0 \cdots \varepsilon_{n-1}})^{(\varepsilon_n)}$ where $\varepsilon_n = -1, 0$, or 1 as in (3.7). Note that $\gamma^{\varepsilon_0 \cdots \varepsilon_n}$ has x-intercept at

$$\frac{1-\rho}{\rho} \sum_{j=0}^{n-1} \varepsilon_{n-j}\, \rho^{-j}. \tag{3.8}$$

Let

$$\Gamma_0 = \{\gamma^{\varepsilon_0 \cdots \varepsilon_n} : n \in \mathbb{N}\},$$
$$\Gamma = \{\gamma \in \Gamma_0 : \gamma \text{ has } x\text{-intercept at } (-1, 1)\}, \text{ and}$$
$$\Gamma^+ = \{\gamma \in \Gamma_0 : \gamma \text{ has } x\text{-intercept at } [0, 1)\}.$$

Since each γ is determined by its x-intercept, by multiplying with $\rho/(1-\rho)$ the expression in (3.8), we have

PROPOSITION 3.4. *Let $\beta = \rho^{-1}$, then $\gamma^{\varepsilon_0 \cdots \varepsilon_n} \in \Gamma_0$ (or Γ) is one to one corresponding to the state $s^{(n)} \in W_{0,\beta}$ (or W_β, respectively) determined by $(\varepsilon_0, ..., \varepsilon_n)$. Moreover T can be consider as a transition operation from $s^{(n)}$ to the three states $s^{(n+1)} = \beta s^{(n)} + \varepsilon_{n+1}$, $\varepsilon_{n+1} = -1, 0, 1$ with weights* 1, 2, 1, *respectively.*

LEMMA 3.5. *T is invariant on $\langle \Gamma_0 \backslash \Gamma \rangle$.*

Proof. The simple proof is essentially the same as in Proposition 2.1. Let $\gamma \in \Gamma_0 \backslash \Gamma$, then γ has x-intercept at $a \geqslant 1$ (or $\leqslant -1$), and $\gamma^{(\varepsilon)}$ has x-intercept $a/\rho + \varepsilon(1-\rho)/\rho$, $\varepsilon = -1, 0$, or 1. That

$$\frac{a}{\rho} + \varepsilon\,\frac{1-\rho}{\rho} \geqslant \frac{1}{\rho} - \frac{1-\rho}{\rho} = 1$$

implies that $T(\gamma) = \gamma^{(-1)} + 2\gamma^{(0)} + \gamma^{(1)} \in \langle \Gamma_0 \backslash \Gamma \rangle$.

Recall that if V is a vector space with $V = V_1 \oplus V_2$, and if $S: V \to V$ is a linear map invariant on V_2, then we can define the quotient map $\tilde{S}: V/V_2 \to V/V_2$. By identifying V/V_2 with V_1, the map $\tilde{S}: V_1 \to V_1$ is given by $\tilde{S}(u) = v$ where $u, v \in V_1$ with $S(u) = v + v'$, $v' \in V_2$. It is easy to show that $(\tilde{S})^n = (S^n)^\sim$. If V is finite dimensional, let $\{e_1, ..., e_m\}$ be a basis of V_1, $\{e_{m+1}, ..., e_n\}$ be a basis of V_2. Let A be the matrix representation of S with respect to the basis $\{e_1, ..., e_n\}$, and let A_1 be the matrix representation of $\tilde{S}$ with respect to the basis $\{e_1, ..., e_m\}$, then

$$A = \begin{bmatrix} A_1 & 0 \\ B_1 & A_2 \end{bmatrix}.$$

It follows from Proposition 3.5 and the above remark that we can define the quotient map $\tilde{T}: \langle \Gamma \rangle \to \langle \Gamma \rangle$. Define also $\tilde{T}^+: \langle \Gamma^+ \rangle \to \langle \Gamma^+ \rangle$ by $\tilde{T}^+ = \pi \circ \tilde{T}$, where π is the projection map of $\langle \Gamma \rangle$ onto $\langle \Gamma^+ \rangle$ given by $\pi(\gamma) = \gamma'$, where γ and γ' have x-intercepts a and $|a|$, respectively.
We use the following proposition repeatedly in Section 4.

PROPOSITION 3.6. *Let $\gamma \in \langle \Gamma^+ \rangle$, then*

(i) $\Phi_\gamma^{(\alpha)}(h) = 0$ *for all* $h > 0$ *implies that* $\gamma = 0$,

(ii) *there exists* $\delta > 0$ *such that*

$$\Phi_\gamma^{(\alpha)}(h) = \frac{1}{4\rho^\alpha} \Phi_{\tilde{T}^+(\gamma)}^{(\alpha)}\left(\frac{h}{\rho}\right) + o(h^\delta), \qquad h > 0,$$

where $o(h^\delta) \geqslant 0$ *is zero order* h^δ *as* $h \to 0$.

Proof. (i) follows from Lemma 3.2. To prove (ii), we use Proposition 3.3 and observe that

$$\Phi_{\tilde{T}^+(\gamma)}^{(\alpha)}(h) = \Phi_{\tilde{T}(\gamma)}^{(\alpha)}(h)$$

(Proposition 3.1(i)), and

$$\Phi_{\tilde{T}(\gamma)}^{(\alpha)}(h) = \Phi_{T(\gamma)}^{(\alpha)}(h) + o(h^\delta)$$

(Proposition 3.1(ii), (iii)).

The reader is referred to Section 5 for some concrete examples of Γ, Γ^+, that are finite sets, and the matrix representation of $\tilde{T}$ and $\tilde{T}^+$ on them.

4. THE THEOREMS

We use some facts concerning eigenvalues of non-negative matrices (Perron–Frobenius Theorem, see, e.g., [Se, V]). Let $A = [a_{ij}]$ with $a_{ij} \geqslant 0$, then A has a positive eigenvalue λ such that $|\lambda'| \leqslant \lambda$ for all other eigenvalues λ' of A, and the corresponding eigenvectors have nonnegative coordinates. Also

$$\min_j \sigma_j \leqslant \lambda \leqslant \max_j \sigma_j, \tag{4.1}$$

where $\sigma_j = \sum_{i=1}^n a_{ij}$. If A is *irreducible*, then λ is simple, the corresponding eigenvector for λ has all coordinates positive, and λ equals the maximum or minimum if and only if $\sigma_1 = \cdots = \sigma_n$. If further A is *aperiodic* (also called primitive, i.e., there exists n such that all the entries of A^n are positive), then $|\lambda'| < \lambda$. A sufficient condition for a non-negative matrix A to be aperiodic is that A is irreducible and some of its diagonal elements are positive [V, p. 43, Ex. 1].

It follows from definition that if $\rho^{-1} = \beta$ is an F-number, then Γ is a finite set, hence $\tilde{T}: \langle \Gamma \rangle \to \langle \Gamma \rangle$, $\tilde{T}^+: \langle \Gamma^+ \rangle \to \langle \Gamma^+ \rangle$ have non-negative matrix representations and the eigenvalues exist.

PROPOSITION 4.1. *Let $\frac{1}{2} < \rho < 1$ such that $\rho^{-1} = \beta$ is an F-number, then the maximal eigenvalues of $\tilde{T}$ and $\tilde{T}^+$ are equal.*

Moreover if $\tilde{T}^+$ is irreducible, let λ be its maximal eigenvalue, and let λ' be the other eigenvalues, then $|\lambda'| < \lambda$.

Proof. It is easy to show that $\tilde{T}^+ \circ \pi = \pi \circ \tilde{T}$. Let λ and λ^+ be the maximal eigenvalues of $\tilde{T}$ and $\tilde{T}^+$, respectively. Suppose $\gamma \in \langle \Gamma \rangle$ is an eigenvector of $\tilde{T}$ corresponding to λ, then $\gamma = \sum_i c_i \gamma_i$, $\gamma_i \in \Gamma$, $c_i \geqslant 0$ but not all 0. This implies that $\pi(\gamma) \neq 0$. Hence

$$\tilde{T}^+(\pi(\gamma)) = \pi(\tilde{T}(\gamma)) = \pi(\lambda\gamma) = \lambda\pi(\gamma)$$

and λ is an eigenvalue of $\tilde{T}^+$, so that $\lambda \leqslant \lambda^+$. On the other hand, taking the adjoint of the identity $\tilde{T}^+ \circ \pi = \pi \circ \tilde{T}$, we have $\pi^* \circ (\tilde{T}^+)^* = \tilde{T}^* \circ \pi^*$. The eigenvalues of $\tilde{T}^*$ and $(\tilde{T}^+)^*$ are unchanged, and the same argument as above implies that $\lambda^+ \leqslant \lambda$.

To prove the second assertion, we need to recall that $T(\gamma^0) = (\gamma^0)^{(-1)} + 2(\gamma^0)^{(0)} + (\gamma^0)^{(1)}$, hence

$$\tilde{T}^+(\gamma^0) = 2(\gamma^0)^{(0)} + 2(\gamma^0)^{(1)} = 2\gamma^0 + 2(\gamma^0)^{(1)}.$$

This implies that the entry corresponding to γ^0 on the diagonal of matrix A representing $\tilde{T}^+$ is not zero. The remark before the proposition implies that A is aperiodic, and the result follows.

Our first main theorem is

THEOREM 4.2. *Let $\frac{1}{2} < \rho < 1$ such that $\rho^{-1} = \beta$ is an F-number, and let μ be the self-similar measure defined by ρ as in* (3.1), *then μ is singular. Moreover,* $\dim_{\text{m.q.v.}}(\mu) = \alpha$ *is given by $4\rho^\alpha = \lambda$, where λ is the maximal eigenvalue of $\tilde{T}^+$ on $\langle \Gamma^+ \rangle$.*

We need a few lemmas.

LEMMA 4.3. *Suppose $\beta = \rho^{-1}$ is an F-number. Let A be the matrix representation of $\tilde{T}^+$ on $\langle \Gamma^+ \rangle$ with respect to the basis Γ^+, and let λ be the maximum eigenvalue of A, then $\lambda < 4$, and $\lambda \neq 4\rho$.*

Proof. Note that the original definition of T on $\gamma \in \Gamma_0$ is defined in (3.6) by $T(\gamma)=\gamma^{(-1)}+2\gamma^{(0)}+\gamma^{(1)}$. $\tilde{T}(\gamma)$ defined on Γ is obtained by truncating the component $\gamma^{(\varepsilon)}$ that has an x-intercept outside $(-1, 1)$. Hence the matrix representation of $\tilde{T}^+$ on $\langle \Gamma^+ \rangle$ with respect to the basis Γ^+ has column sums σ_j equal to either 4, 3, or 1, and not all σ_j are equal. It follows from the remark in the beginning of this section that $\lambda < 4$ (consider the irreducible component corresponding to λ).

To show that $\lambda \neq 4\rho$, we assume the contrary; suppose 4ρ is the maximal eigenvalue of A. Let $q(x)$ be the characteristic polynomial of A, it has integer coefficients. Let $p(x)=\sum_{j=1}^{d} a_j x^j$ be the minimal polynomial of $\beta=\rho^{-1}$, and let β' denote conjugate roots of β, then 4ρ, $4\beta'^{-1}$ satisfies $\tilde{p}(x)=\sum_{j=1}^{d} 4^j a_j x^{d-j}$. It follows that $\tilde{p}(x)$ divides $q(x)$, so that both 4ρ, $4\beta'^{-1}$ are roots of $q(x)$, hence eigenvalue of A. On the other hand β is an F-number, Theorem 2.6 implies that $|\beta'| < \beta$ so that $|4\beta'^{-1}| > 4\rho$. This contradicts that 4ρ is the maximal eigenvalue of A, and the proof is completed.

LEMMA 4.4. *Under the same hypotheses and notations of Theorem* 4.2, *then for any* $0 < \beta < \alpha$,

$$\limsup_{h \to 0} \frac{1}{h^{1+\beta}} \int_{-\infty}^{\infty} |\mu(Q_h(t))|^2 = 0.$$

Proof. We prove the lemma by considering the following two cases:

(i) A is irreducible: Let γ be an eigenvector corresponding to λ, then $\gamma = \sum_{\gamma_i \in \Gamma^+} c_i \gamma_i$ where $c_i > 0$ for all such i. By Proposition 3.6,

$$\Phi_\gamma^{(\beta)}(h) = \frac{1}{4\rho^\beta} \Phi_{\tilde{T}^+\gamma}^{(\beta)}\left(\frac{h}{\rho}\right) + o(h^\delta) \leqslant \frac{\lambda}{4\rho^\beta} \Phi_\gamma^{(\beta)}\left(\frac{h}{\rho}\right) + o(h^\delta).$$

A direct computation shows that

$$\Phi_\gamma^{(\beta)}(\rho^n h) \leqslant \left(\frac{\lambda}{4\rho^\beta}\right)^n \Phi_\gamma^{(\beta)}(h) + \sum_{j=1}^{n} \left(\frac{\lambda}{4\rho^\beta}\right)^{n-j} o(\rho^j h)^\delta.$$

This implies that for $\max\{\lambda/4\rho^\beta, \rho^\delta\} < b < 1$,

$$\Phi_\gamma^{(\beta)}(\rho^n h) \leqslant b^n(\Phi_\gamma^{(\beta)}(h) + o(h^\delta)), \tag{4.2}$$

hence $\lim_{h \to 0} \Phi_\gamma^{(\beta)}(h) = 0$. The positivity of c_j corresponding to $\gamma_j = \gamma^0$ implies that

$$0 = \lim_{h \to 0} \Phi_{\gamma^0}^{(\beta)}(h) = \lim_{h \to 0} \frac{1}{h^{1+\beta}} \int_{-\infty}^{\infty} |\mu(Q_h(t))|^2.$$

(ii) A is reducible: By rearranging the basis elements, we can assume that

$$A = \begin{bmatrix} A_k & 0 & \cdot & \cdot & \cdot & 0 \\ & \cdot & 0 & & & \cdot \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & & & & A_2 & 0 \\ & & & & & A_1 \end{bmatrix},$$

where A is zero on the upper right triangle, and each A_k is irreducible. The assumption implies that all eigenvalues are not greater than $\lambda = 4\rho^{\alpha}$. Let V_j be the corresponding subspace of A_j. The same argument as in (i) applied on $\tilde{T}_1^+ : V_1 \to V_1$, the restriction of $\tilde{T}^+$ on V_1, implies that there exists an eigenvector $\gamma = \sum_i c_i \gamma_i$, with $c_i > 0$, $\gamma_i \in \Gamma^+ \cap V_1$ such that (4.2) holds. The positivity of c_i further implies that there exists $0 < b < 1$,

$$\Phi_{\gamma_i}^{(\beta)}(\rho^n h) \leqslant b^n(\Phi_{\gamma_i}^{(\beta)}(h) + o(h^{\delta})), \qquad \gamma_i \in \Gamma^+ \cap V_1, \tag{4.3}$$

so that

$$\Phi_{\gamma_i}^{(\beta)}(h) = o(h^{\eta_1}), \qquad \gamma_i \in \Gamma^+ \cap V_1, \tag{4.4}$$

where $\eta_1 = \ln b / \ln \rho > 0$. Assume (4.3) and (4.4) hold for $V_1, ..., V_{j-1}$. Let $\tilde{T}_j^+ : V_j \to V_j$ be the induced map with representation A_j. Let γ be the eigenvector of $\tilde{T}_j^+$ on V_j with $\gamma = \sum_i c_i \gamma_i$, $c_i > 0$, $\gamma_i \in \Gamma^+ \cap V_j$. Note that $\tilde{T}^+(\gamma) = \tilde{T}_j^+(\gamma) + \gamma'$, where $\tilde{T}_j^+(\gamma) \in V_j$, $\gamma' \in V_{j-1} \oplus \cdots \oplus V_1$. Hence

$$\begin{aligned} \Phi_{\gamma}^{(\beta)}(h) &= \frac{1}{4\rho^{\beta}} \Phi_{\tilde{T}^+\gamma}^{(\beta)}\left(\frac{h}{\rho}\right) + o(h^{\delta}) = \frac{1}{4\rho^{\beta}} \Phi_{\tilde{T}_j^+(\gamma)+\gamma'}^{(\beta)}\left(\frac{h}{\rho}\right) + o(h^{\delta}) \\ &\leqslant \frac{\lambda}{4\rho^{\beta}} \Phi_{\gamma}^{(\beta)}\left(\frac{h}{\rho}\right) + \frac{1}{4\rho^{\beta}} \Phi_{\gamma'}^{(\beta)}\left(\frac{h}{\rho}\right) + o(h^{\delta}). \end{aligned}$$

The same iteration argument implies that there exists $0 < b < 1$,

$$\Phi_{\gamma}^{(\beta)}(\rho^n h) \leqslant b^n(\Phi_{\gamma}^{(\beta)}(h) + \Phi_{\gamma'}^{(\beta)}(h) + o(h^{\delta})), \qquad \gamma \in \Gamma^+ \cap V_j,$$

this implies that $\Phi_{\gamma}^{(\beta)}(h) = o(h^{\eta_j})$, $\eta_j > 0$, $\gamma \in \Gamma^+ \cap V_j$. It follows from induction that

$$\lim_{h \to 0} \Phi_{\gamma}^{(\beta)}(h) = 0, \qquad \gamma \in \Gamma^+ \cap V_j, \quad j = 1, ..., k.$$

One such γ must equal γ^0, and the same conclusion as that in (i) follows.

COROLLARY 4.5. *Under the same hypotheses and notations as in Theorem* 4.2, *then* $4\rho < \lambda < 4$.

Proof. In view of Lemma 4.3, we need only show that $\lambda \not< 4\rho$. Suppose $\lambda < 4\rho$, then $\lambda = 4\rho^{\alpha}$ with $\alpha > 1$. By taking $\beta = 1$ in Lemma 4.4, we have

$$\limsup_{h \to 0} \frac{1}{h^2} \int_{-\infty}^{\infty} |\mu(Q_h(t))|^2 = 0.$$

On the other hand, the limit supremum implies that

$$\sup_{h > 0} \frac{1}{h^2} \int_{-\infty}^{\infty} |\mu(Q_h(t))|^2 < \infty.$$

By [HL] μ is absolutely continuous and

$$\lim_{h \to 0} \frac{1}{h^2} \int_{-\infty}^{\infty} |\mu(Q_h(t))|^2 = \int_{-\infty}^{\infty} \left(\frac{d\mu}{dx}\right)^2 dx > 0,$$

a contradiction.

Proof of Theorem 4.2. It follows from the above corollary that $4\rho < \lambda < 4$. Hence there exists $0 < \alpha < 1$ and an eigenvector $\gamma \in \langle \Gamma^+ \rangle$ of $\tilde{T}^+$ such that

$$\Phi_{\gamma}^{(\alpha)}(h) = \frac{1}{4\rho^{\alpha}} \Phi_{\tilde{T}^+\gamma}^{(\alpha)}\left(\frac{h}{\rho}\right) + o(h^{\delta}) = \frac{\lambda}{4\rho^{\alpha}} \Phi_{\gamma}^{(\alpha)}\left(\frac{h}{\rho}\right) + o(h^{\delta}) = \Phi_{\gamma}^{(\alpha)}\left(\frac{h}{\rho}\right) + o(h^{\delta}).$$

This implies that $\Phi_{\gamma}^{(\alpha)}(\rho h) = \Phi_{\gamma}^{(\alpha)}(h) + o(h^{\delta})$, so that

$$\Phi_{\gamma}^{(\alpha)}(\rho^n h) = \Phi_{\gamma}^{(\alpha)}(h) + \sum_{j=1}^{n} o((\rho^{j-1}h)^{\delta}).$$

The convergence of $\sum_{j=1}^{\infty} o((\rho^{j-1}h)^{\delta})$ implies that, for each h, the limit $\varphi(h) = \lim_{h \to 0} \Phi_{\gamma}^{(\alpha)}(\rho^n h)$ exists. That λ is maximal implies that $\gamma = \sum_i c_i \gamma_i$, $\gamma_i \in \Gamma_2^+$, $c_i \geq 0$. It follows that $\Phi_{\gamma}^{(\alpha)}(h) = \sum_i c_i \Phi_{\gamma}^{(\alpha)}(h) \geq 0$, so that φ is non-zero. φ is also multiplicatively periodic, i.e., $\varphi(\rho h) = \varphi(h)$, and hence bounded. We conclude that

$$\Phi_{\gamma}^{(\alpha)}(h) = \varphi(h) + o(h^{\delta}), \tag{4.5}$$

and

$$0 < \limsup_{h \to 0} \Phi_{\gamma}^{(\alpha)}(h) = \limsup_{h \to 0} \frac{1}{h^{1+\alpha}} \int_{\gamma} \mu(Q_h(x))\, \mu(Q_h(y)) < \infty. \tag{4.6}$$

That $\gamma = \sum_i c_i \gamma_i$, $c_i \geqslant 0$, $\gamma_i \in \Gamma^+$ also implies that there exists i such that

$$0 < \limsup_{h \to 0} \frac{1}{h^{1+\alpha}} \int_{\gamma_i} \mu(Q_h(x))\, \mu(Q_h(y)) \leqslant \limsup_{h \to 0} \frac{1}{h^{1+\alpha}} \int_{-\infty}^{\infty} |\mu(Q_h(t))|^2.$$

It follows that $\dim_{\text{m.q.v.}}(\mu) \leqslant \alpha$ (<1). On the other hand, for any $\beta < \alpha$, Lemma 4.3 implies that

$$\limsup_{h \to 0} \frac{1}{h^{1+\beta}} \int_{-\infty}^{\infty} |\mu(Q_h(t))|^2 = 0,$$

so that $\dim_{\text{m.q.v.}}(\mu) \geqslant \alpha$.

Erdős and Salem proved that for $1 < \beta < 2$, β is a P.V. number if and only if the Fourier transformation of the self-similar measure μ defined by $\rho = \beta^{-1}$ satisfies $\hat{\mu}(\xi) \nrightarrow 0$ as $\xi \to \infty$ [S]. In the following we consider the mean quadratic average of the Fourier transformation of such measures corresponding to the F-numbers. We need the following theorem which is a special case of [LW, Corollary 4.12]:

THEOREM 4.6. *Let μ be a bounded regular Borel measure on $\mathbb{R}$, then the following two statements are equivalent*:

(i) $\lim_{h \to 0}((1/h^{1+\alpha}) \int_{-\infty}^{\infty} |\mu(Q_h(t))|^2 - \varphi(h)) = 0$ *for some multiplicative periodic function φ of period ρ.*

(ii) $\lim_{r \to \infty}((1/r^{1-\alpha}) \int_{-r}^{r} |\hat{\mu}|^2 - \phi(r)) = 0$ *for some multiplicative period function ϕ of period ρ.*

Note that in the proof of Theorem 4.2, we have shown that

$$\Phi_{\gamma}^{(\alpha)}(h) = \varphi(h) + o(h^{\delta})$$

for the special eigenvector γ (see (4.5)). Statement (i) is the case with $\gamma = \gamma^0$, and we prove it in the following theorem under an additional hypothesis:

THEOREM 4.7. *Under the same hypotheses of Theorem* 4.2, *let* $\alpha = \dim_{\text{m.q.v.}}(\mu)$. *If A is irreducible, then*

$$\Phi^{(\alpha)}(h) = \varphi(h) + o(h^{\eta}),$$

for some φ that is non-zero, bounded, and has multiplicative period ρ.

Proof. Note that A is irreducible, the c_i in the eigenvector $\gamma = \sum_i c_i \gamma_i$ in (4.6) are all positive, and one of the γ_i equals γ^0. Hence (4.6) implies that

$$0 < \limsup_{h \to 0} \Phi^{(\alpha)}(h) = \limsup_{h \to 0} \frac{1}{h^{1+\alpha}} \int_{\gamma^0} \mu(Q_h(x))\, \mu(Q_h(y)) < \infty,$$

and $\Phi^{(\alpha)}$ is bounded.

Let $\lambda_1 = 4\rho^\alpha, \lambda_2, ..., \lambda_k$ be the eigenvalues of A. Since A is irreducible, λ_1 is a simple eigenvalue and by Proposition 4.1, $|\lambda_i| < \lambda_1$ for $i = 2, ..., k$. Let γ_1 be the eigenvector corresponding to λ_1, and let $\{\gamma_{ij}\}$ be a Jordan basis of the eigensubspace of λ_i for $i = 2, ..., k$; i.e.,

$$\tilde{T}^+(\gamma_1) = \lambda_1 \gamma_1, \qquad \tilde{T}^+(\gamma_{i(j+1)}) = \lambda_i \gamma_{i(j+1)} + \gamma_{ij}.$$

That $\Phi^{(\alpha)}(h)$ is bounded implies that $\Phi^{(\alpha)}_{\gamma_{ij}}(h)$ is also bounded. A repeated application of Proposition 3.6(ii) yields

$$\Phi^{(\alpha)}_{\gamma_{ij}}(h) = O\left(\left(\frac{\lambda}{4\rho^\alpha}\right)^n\right) + o(h^\delta), \qquad h > 0,$$

where n is such that $\rho^{n+1} < h$, $\rho^n \geqslant h$; i.e., $n = [\ln h / \ln \rho]$. This implies that $\Phi^{(\alpha)}_{\gamma_{ij}}(h)$ is of small order h^η for all i and j as $h \to 0$. Now if we write

$$\gamma^0 = a_1 \gamma_1 + \sum_i \sum_j a_{ij} \gamma_{ij},$$

then

$$\Phi^{(\alpha)}(h) := \Phi^{(\alpha)}_{\gamma^0}(h) = a_1 \Phi^{(\alpha)}_{\gamma_1}(h) + \sum_i \sum_j a_{ij} \Phi^{(\alpha)}_{\gamma_{ij}}(h) = \varphi(h) + o(h^\eta),$$

where φ is a multiplicative periodic function of period ρ and $0 < \eta < \delta$.

Theorems 4.6 and 4.7 yield

THEOREM 4.8. *Let $\frac{1}{2} < \rho < 1$ such that $\rho^{-1} = \beta$ is an F-number, and let μ be the self-similar measure defined by ρ as in* (3.1). *Let $\alpha = \dim_{\text{m.q.v.}}(\mu)$, and suppose A is irreducible, then*

$$\frac{1}{r^{1-\alpha}} \int_{-r}^{r} |\hat{\mu}|^2 = \phi(r) + o(1),$$

where $o(1) \to 0$ as $r \to \infty$, ϕ is non-zero, bounded, and has multiplicative period ρ.

5. Examples and Remarks

Example 5.1. Let $\rho=(\sqrt{5}-1)/2$, then $\beta=(\sqrt{5}+1)/2$. By using $\rho^2+\rho-1=0$, it is easy to show that $\Gamma=\{\gamma:\gamma$ has x-intercept at $a=0, \rho, \rho^2, -\rho, -\rho^2\}$, $\Gamma^+=\{\gamma:\gamma$ has x-intercept at $a=0, \rho, \rho^2\}$. The matrix representation of $\tilde{T}$ and $\tilde{T}^+$ are

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix},$$

respectively. The corresponding auxiliary sets W_β and W_β^+ are $\{0, 1, \rho, -1, -\rho\}$ and $\{0, 1, \rho\}$, respectively. The characteristic polynomial of $\tilde{T}^+$ is x^3-2x^2-2x+2. The maximal eigenvalue λ can hence be calculated. By equating $4\rho^\alpha=\lambda$, we find that the m.q.v. dimension of the corresponding self-similar measure is $\alpha=0.9923994...$. This result is also obtained in [L] by using a different method.

Example 5.2. The following is a list of P.V. numbers that the m.q.v. dimensions have been calculated by using Theorem 4.2. It was provided by M. F. Ma:

Min. Polynomial	β	ρ	Size of A	λ	$\dim_{\text{m.q.v.}}(\mu_\rho)$
$x^3-x^2-x-1=0$	1.8392868	0.54368899	4	2.2226941	0.9642200
$x^3-2x^2+x-1=0$	1.7548777	0.56984028	7	2.2941040	0.9885364
$x^2-x-1=0$	1.6180340	0.61803340	3	2.4811943	0.9923994
$x^3-x^2-1=0$	1.4655719	0.68232750	25	2.7302333	0.9991163
$x^4-x^3-1=0$	1.3802776	0.72449194	627	2.8979776	0.9999895
$x^3-x-1=0$	1.3247180	0.75487764	90	3.0195190	0.9999901

Note that the β corresponding to $x^3-x-1=0$ is the smallest P.V. number [Si]. There are only one P.V. number of degree 2 and four of degree 3; they are all listed here. We have not been able to show that the matrix A representing $\tilde{T}^+$ is irreducible in general. However, all the examples above indicate that this is true, and Theorems 4.7 and 4.8 apply to the above cases.

The above chart is listed according to decreasing order of β; i.e., the increasing order of ρ. A very interesting observation is that as ρ increases, the dimension increases to 1.

The relationships of the F-numbers with the two well known classes of algebraic numbers can be summarized as (Theorem 2.5, Proposition 2.6)

$$\text{P.V. numbers} \subseteq F\text{-numbers} \subseteq \text{Beta numbers}.$$

So far we have not been able to prove that either one of the inclusions is proper. However, numerical evidence shows that the second inclusion should be proper. Recall that β is called a *Salem number* if $\beta > 1$ is an algebraic integer and $|\beta'| \leqslant 1$ for all its conjugate roots, with at least one satisfying $|\beta'| = 1$. We do not know how the Salem numbers relate to the F-numbers. It is still an open question whether Salem numbers are Beta numbers ([Bo, Sc]).

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